Varsity Team Ciphering Solutions

1) **(The answer is:)**

2) **(16/63)** Rearrange the equation and integrate:

$$\int \frac{f''(f^{-1}(x))}{f'(f^{-1}(x))}\,dx = \int \frac{-f'(x)}{[f(x)]^2}\cdot f''\left(\frac{1}{f(x)}\right)\,dx$$

$$f'(f^{-1}(x)) = f'\left(\frac{1}{f(x)}\right)$$

Since $f'(x)$ is one-to-one, this implies that $f^{-1} = \frac{1}{f}$. Since $f(7) = 9$, $f^{-1}(7) = 1/9$, and since $f^{-1}(9) = 7$, $f(9) = 1/7$. Their sum is $1/9 + 1/7 = 16/63$.

3) **(5913)** Note that

$$\sum_{k\geq 1} kx^k = \frac{x}{(1-x)^2}$$

Differentiate and multiply by x repeatedly to get:

$$\sum_{k\geq 1} k^2x^{k-1} = \frac{x+1}{(1-x)^3}$$

$$\sum_{k\geq 1} k^2x^k = \frac{x(x+1)}{(1-x)^3}$$

$$\sum_{k\geq 1} k^3x^{k-1} = \frac{x^2+4x+1}{(x-1)^4}$$

$$\sum_{k\geq 1} k^3x^k = \frac{x(x^2+4x+1)}{(x-1)^4}$$

Let $x = 1/11$ to find the sum to be 913/5000, so that $a + b = 5913$.

4) **(91) CORRECTION: THE ORIGINAL PROBLEM SHOULD HAVE** $4321x + 4545 \equiv 9340$.
First, simplify the congruences:

$$\begin{aligned} 10x + 3 &\equiv 1 \pmod{24} \\ x + 1 &\equiv 12 \pmod{16} \end{aligned}$$

From the first congruence, $x = 12m + 7$, for any integer m. From the second congruence, $x = 16j + 11$, for any integer j. Set these two expressions for x equal and consider them (mod 16). This gives $m = 4k + 3$ for any integer k. Substitute this expression for m to get $x = 48k + 43$ so that $a + b = 91$.

5) **(979)** Let A_n be the event 'have the disease after test n' and B_n be the event 'test positive on test n'. We know that $P(A_0) = 1/2$, $P(A_1) = (1/2)(1/2) + 1/2 = 3/4$, and $P(A_2) = (1/2)(1/2)(1/3) + 1/2 + 1/4 = 5/6$. Then we have $P(B_0) = (1/2)(1/5) + (1/2)(4/5) = 1/2$, $P(B_1) = (1/2)(4/5)+(1/2)(1/(1+1))(4/5)+(1/2)(1/2)(1/5) = 13/20$, and $P(B_2) = (1/2)(4/5)+(1/2)(1/(2))(4/5) + (1/2)(1/3)(4/5) + (1/2)(2/3)(1/5) = 4/5$.
Conditional probability says that

$$P(A_2)P(B_0 \cap B_1 \cap B_2|A_2) = P(B_0 \cap B_1 \cap B_2)P(A_2|B_0 \cap B_1 \cap B_2)$$

We know that $P(A_2) = 5/6$, $P(B_0 \cap B_1 \cap B_2|A_2) = P(B_2|A_2) = 4/5$, and $P(B_0 \cap B_1 \cap B_2) = 1/2 + 13/20 + 4/5 - (1/2)(13/20) - (1/2)(4/5) - (13/20)(4/5) + (1/2)(13/20)(4/5) = 193/200$. Therefore, $P(A_2|B_0 \cap B_1 \cap B_2) = 400/579$. So $a + b = 979$.

6) **($a^2b^2/4$)** This is true in general, but to find F, you can be cheap and consider the right triangle. The formula becomes

$$\frac{ab}{2} = \sqrt{F(a, b, c)} \Longrightarrow F(a, b, c) = a^2b^2/4$$

7) **(2)** Add the equations and rearrange to get

$$31 = x^5 + 10x^3y^2 + 5xy^4$$

and then subtract the second equation from the first to get

$$1 = 5x^4y + 10x^2y^3 + y^5$$

Combine these to get $(x + y)^5 = 32 \Rightarrow x + y = 2$.

8) **(46)** The area A is given by

$$K = \frac{1}{4}\sqrt{(1 + r + r^2)(1 + r - r^2)(1 - r + r^2)(-1 + r + r^2)}$$

Equating this with the perimeter, $1 + r + r^2$, and rearranging gives

$$r^8 + 15r^4 + 32r^3 + 46x^2 + 32x + 17 = 2r^6$$

so that $K = 46$.

9) **(Vandermonde)** Just gotta know it. Send your thanks to Lassiter for creating a "cheap question" precedent. Specifically, the identity is the "Vandermonde Convolution" identity.

10) **($\sigma\sqrt{2}$)** To find $\sigma_Y = \sigma_{X+X}$, use the formula $\sigma_{A+B} = \sqrt{\sigma_A^2 + \sigma_B^2}$. This gives $\sigma_Y = \sigma\sqrt{2}$.