Milton Math Tournament

*Varsity Individual Written Test Solutions*

## PART I: Multiple Choice

1) **(B)** As $x \to a$ from the right, $g$ must approach $a$ as well (by the squeeze theorem). However, on the interval, $g \neq a$, so $g$ must change value at least once, and thus cannot be constant.

2) **(B)** Multiply both the numerator and denominator by $(\sin 2x)(\cos 4x)$. Simplifying $f$ yields $f(x) = \sin 2x \cos 4x$, so that the period is simply $\pi/2$. (note that cosine dominates the period since it cycles less often).

3) **(E - 134)** Let $b = n$. This gives $a\S n = a^2 - n\S n$. Substitute this in the original definition to get $a\S b = a^2 - (b^2 - n\S n)$. Therefore, for $a = b$, $a\S b = n\S n$. Since $16\S 16 = 14$, For all $a, b$, we have

$$a\S b = a^2 - b^2 + 14$$

Letting $a = 13$ and $b = 7$ yields 134.

4) **(C)**

5) **(A)** Use the identity $\sin(90 - \theta) = \cos\theta$ to simplify the equation to $\sin 7\theta = \sin(90 - 13\theta)$. To find the smallest $\theta$, $7\theta$ and $90 - 13\theta$ can be assumed to be acute, so that the arguments must be equivalent:

$$7\theta = 90 - 13\theta \ \Rightarrow \theta = 4.5$$

6) **(D)** Firstly, (4) is trivial. Also, using a right triangle whose altitude divides its hypotenuse into pieces of length $a$ and $b$, we find that the altitude has length $\sqrt{ab}$. Thus (2) is possible. For (3), consider the following: use the segment of length 1 to create a segment of length 2. Create an arbitrary circle, and make the new segment a chord of the circle. Any chord that intersects the midpoint of this chord will have itself split into lengths of $x$ and $1/x$ (by power of a point). Simply use the segment of length $a$ to get the desired length. So (3) is possible. Via a similar method, one could construct a length of $ab$, but a length of 1 would need to be given. Since no length of 1 is given, construction (1) is impossible. (note that the units of $ab$ are length squared—division by another segment of length 1 is required to bring the units to length, much like the $1/a$ example actually is (1 unit)(1 unit)/($a$ units)).

7) **(B)** Define $f(x) = P(x) - Q(x)$. Since $P = Q$ for $x = 1, 2, 3, 4, 5$ and $P$ and $Q$ both have degree 5, $f$ can be written as $f(x) = a(x-1)(x-2)(x-3)(x-4)(x-5)$. Since $P(0) = Q(0) - 300$, $f(0) = -300 \Rightarrow a = 15/6$. Now we have $P(7) = f(7) + Q(7) = 1800 - 1234 = 566$.

8) **(C)** If choice A were right, there would be no answer, which implies that even A is wrong! So throw that out. If D were the answer, that would imply that C, E, and even D itself are wrong. Another contradiction; throw D out. Choice C works fine, since it says that A is wrong (which is right) and B is wrong, which must be true for C to be right. Thus C works. Note that, at this point, either one or two choices can work (other than E). If B is chosen, it negates E, which makes it so that two above E do *not* work, which is wrong when B is right (C works, as we have shown). Throw B out. Now, since only one choice above E works, throw E out. C is the only answer.

9) **(B)** The law of sines states that

$$\frac{AB}{\sin\angle C} = \frac{BC}{\sin\angle A} \Rightarrow \frac{\sqrt{\sin\theta}}{\sin\left(\frac{180-\theta}{2}\right)} = \frac{\sin\theta}{\sin\theta} = 1$$

This rearranges into

$$\sin^2(90-\theta/2) = \sin\theta \Rightarrow \frac{1+\cos\theta}{2} = \sin\theta \Rightarrow 5\sin^2\theta - 4\sin\theta = 0$$

so that $\sin\theta = 4/5$. Using the area formula $K = \frac{1}{2}ab\sin C$ with angle $A$ yields

$$K = \frac{1}{2}(\sqrt{\sin\theta})(\sqrt{\sin\theta})(\sin\theta) = \frac{1}{2}\sin^2\theta = 8/25$$

10) **(A)** 2003 is prime, so $d(2003^n) = n+1$. Since $n = 2003$, this reduces the expression to $d(2004) = d(2^2 \cdot 3 \cdot 167) = (2+1)(1+1)(1+1) = 12$.

11) **(A)**

12) **(E - 5100.5)** The sequence $a_k$ is given by $a_k = k + 1/2$. The sum is 5100.5.

13) **(C)** The innermost sum, when expanded has $j-k+1$ terms. Thus, by the second sum, $k-k+1$ terms are added to $(k+1)-k+1$ terms, and so on, up to $k^2-k+1$ terms. The total number of terms is

$$1+2+3+\cdots+(k^2-k+1) = \sum_{j=k}^{k^2} j-k+1 = \sum_{j=k}^{k^2}\left\langle \sum_{m=k}^{j} k^{j^m} \right\rangle$$

This last equality implies that in general (and it can be proven),

$$\left\langle \sum\sum\sum\cdots\sum a_k \right\rangle = \sum\sum\sum\cdots\sum\left\langle \sum a_k \right\rangle$$

So the problem can be reduced to

$$\left\langle \sum_{k=1}^{5}\sum_{j=k}^{k^2}\sum_{m=k}^{j} k^{j^m} \right\rangle = \sum_{k=1}^{5}\sum_{j=k}^{k^2}\left\langle \sum_{m=k}^{j} k^{j^m} \right\rangle$$
$$= \sum_{k=1}^{5}\sum_{j=k}^{k^2}(j-k+1)$$

Evaluating this sum yields

$$\left\langle \sum_{k=1}^{5}\sum_{j=k}^{k^2}\sum_{m=k}^{j} k^{j^m} \right\rangle = \sum_{k=1}^{5}\sum_{j=k}^{k^2}(j-k+1) = 357$$

14) **(C)** The general form for an ellipse is $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$. Dividing this by $A$ gives $x^2+Bxy/A+Cy^2/A+Dx/A+Ey/A+F/A=0$. It takes 5 points to uniquely determine the coefficients $B/A$, and so on. Note that for a circle, the equation reduces to $x^2+y^2+Ax+By+C=0$ (no $xy$ term is needed). Thus, for a circle, only three points are needed (a well-known fact).

15) **(E - 24)** Since Bob works at a constant rate, he paints the fence at a rate of $1/12$ per hour. Let $J$ be the number of hours Joe would take to paint the fence when working at his regular rate constantly. Then during his regular time, he paints $1/J$ of the fence every hour; during the slow time, his rate is $1/2J$, and in his fast time, it is $2/J$. In 8 hours, Joe's work with Bob can be described symbolically:

$$3\frac{1}{J} + \frac{1}{2J} + \frac{2}{J} + 3\frac{1}{J} + 8\frac{1}{12} = 1 \Rightarrow J = 51/2$$

So working alone,

$$\left(3(2/51) + (1/51) + (4/51)\right) \cdot 4 + 3(2/51) + 1(1/51) = 1$$

Which implies that Joe takes 24 hours by himself (4 complete cycles and another cycle, without the fast working stage).

16) **(D)** Reflect the point $(5, 2)$ over the $x$ and $y$ axes to get $(-5, -2)$. By properties of reflection, the straight-line distance between $(2, 4)$ and $(-5, -2)$ is the required distance. Therefore the distance $d$ is

$$d = \sqrt{(2 - (-5))^2 + (4 - (-2))^2} = \sqrt{85}$$

17) **(C)** The problem asks to find different ways of choosing $r$ objects from $m + n + p$. One way is consider choosing $k$ from $m$ of the objects, $j$ from $n$ of the objects, and however many more choices you need to make $r$ from the final $p$ objects. Then, add all of the combinations of $j$ and $k$. Symbolically, this translates into I. II can be rewritten, using the fact that $\binom{n}{r} = \binom{n}{n-r}$:

$$\sum_{k=0}^{r} \binom{m}{k} \binom{n+p}{r-k}$$

This is equivalent to $\binom{m+n+p}{r}$ by Vandermonde's convolution identity.
The third one is completely made up.

18) **(B)** Consider the matrices $BABAB$ and $ABABA$. From the given information they both equal $0_{n\times n}$. If $B$ is invertible, then $BABAB = 0_{n\times n} \Rightarrow BABABB^{-1} = BABA = 0_{n\times n}B^{-1} = 0_{n\times n}$. Likewise, if $A$ is invertible, $ABABA = 0_{n\times n} \Rightarrow A^{-1}ABABA = BABA = A^{-1}0_{n\times n} = 0_{n\times n}$. Thus if either $A$ or $B$ are invertible, $BABA = 0_{n\times n}$.

19) **(D)** We can express $P_n$ as $P_n = a(a+3)(a+6)(a+9) = (a^2 + 9a + 9)^2 - 81$, where $a = a_n$. There are obviously finitely many squares in $P_n$, since for large enough $m$, $m^2 - 81 > (m-1)^2$ implies that eventually, squares can no longer be obtained. And since $P_n = 1, 4, 9$ all have (admittedly complicated) solutions, more than two squares can appear.

20) **(B)** Let $n$ be between the perfect squares $m^2$ and $(m+1)^2$. Then $\sqrt{n}$ is between the consecutive integers $m$ and $m+1$. Thus, $\lfloor\sqrt{n}\rfloor = m$ so that $\lfloor\sqrt{n}\rfloor^2 = m^2$.

21) **(D)** Focus first on the factor on the right.

$$\ln\left(\frac{k^{1/k}}{(k+1)^{1/(k+1)}}\right) = \frac{1}{k}\ln k - \frac{1}{k+1}\ln(k+1) = \frac{1}{k+1}\left[\ln\left(1 - \frac{1}{k+1}\right) + \frac{1}{k}\ln k\right]$$

Thus a simpler form of $F(k)$ is $F(k) = (k+1)^{-1}$. $F(100) = 1/101$.

22) **(C)** Consider the graphs of $f(x) = |x|$ and $g(y) = |y-1|-1$. Then the expression can be seen as the square of the minimum distance from $f$ to $g$. The smallest distance is vertex to vertex (as a graph of the two functions shows), which is $\sqrt{2}$. Thus the minimum value of the original expression is 2.

23) **(A)** Let $x = y = 1$ to find the sum of the coefficients. $(-2)^6 = 64$.

24) **(B)** Each integer, from 2 through 13 can be either included or excluded. That is 2 choices for each of 12 objects: $2^12 = 4096$.

25) **(D)** For the geometric sequence $a, ar, ar^2, \ldots$, we can express $P_n$ as follows:

$$P_n = a(ar)(ar^2)(\cdots)(ar^{n-1}) = a^n r^{0+1+2+\cdots+(n-1)} = a^n r^{n(n-1)/2}$$

So the sum, which we call $S$, is

$$S = \sum_{n=1}^{\infty} \sqrt[n]{P_n} = \sum_{n=1}^{\infty} \sqrt[n]{a^n r^{n(n-1)/2}} = \sum_{n=1}^{\infty} ar^{(n-1)/2}$$

This sum can be rearranged into a geometric series:

$$\sum_{n=1}^{\infty} ar^{(n-1)/2} = a\sum_{n=0}^{\infty} (\sqrt{r})^n = \frac{a}{1-\sqrt{r}}$$

Since for the given sequence, $(a, r) = (16, 1/4)$, the sum is

$$S = \frac{16}{1-1/2} = 32$$

26) **(C)** Let $a = 9x^2 - 16$ and $b = 16x^2 - 9$. This reduces the equation to $a^3 + b^3 = (a+b)^3$. Thus either one of $a, b, a+b$ is 0. The values of $x$ are 3/4, -3/4, 4/3, -4/3, 1, -1. The sum is 37/6.

27) **(E - $n^m$)**

28) **(E - $2-\sqrt{3}$)**

29) **(B)**

30) **(A)**

31) **(C)** The probability of happiness at a particular amount is a random variable $P$. The required ‘probability’ is actually the expected value of $P$. The expected value of of $P$ is

$$E(P) = \int_0^{10} \left(\frac{1}{x}\right)\left(\frac{x^5}{10^5}\right) dx = \frac{1}{5}$$

32) **(B)** For $c \geq 0$, $g(c) = c^2 - c + 1/3$. For $c \geq -1$, $g(c) = c^2 + 3c + 7/3$. For $-1 < c < 0$, $g(c)$ is

$$\int_0^{c+1} (x-c)^2\,dx + \int_{c+1}^{1} (x-c-2)^2\,dx = -c^2 - c + 1/3$$

The minimum value of $g(c)$ is 1/12.

33) **(A)** The first equation gives $(\sin x)^{1-y} = \cos x$. The second equation gives $\sin z = \cos^2 x$. Thus $z = 2 - 2y$.

34) **(E - 0)** The curvature is very small, since the radius of curvature is greater than 2 (as an glance at the graph will show). Thus it is closest to 0. The actual value is $5^{-3/2} = 0.089$.

35) **(E - 6)** Note that the header of the function updateSum reads

void updateSum(int day, int number, int sum)

instead of

void updateSum(int day, int number, int & sum)

Due to this discrepancy, the value stored in sum will always be 0, which is the correct value for $n = 1, 2, 5, 10$. Thus 6 values will return the incorrect result.

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PART II : Free Response

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1) **(9/16)** In a truth table, 9 of the 16 rows are true.

2) **(63)** The function for the sequence is $f(n) = \lceil -1/2 + (1/2)\sqrt{1+8n} \rceil$. $f(2003) = 63$.

3) **(Any integral value of $r \geq 10$)** Disregard this question.

4) **(23)** Consider the expression (mod 7). It reduces to $(a+2)x$. If this is congruent to 0 for all $x$, then $a \equiv 3$, so that, for any integer $m$, $a = 5m + 3$. Similarly, we find that $a = 7j + 2$. Combining these and considering them (mod 5) gives $j \equiv 3$ so that $j = 5k + 3$, which can be substituted into $a = 7j + 2$ to get $a = 35k + 23$ for any integer $k$. Obviously, the smallest positive number $a$ is 23.

5) **(4)** Use sum to product formulas.

6) **(2720/9)** Since $(x+y)^3 = x^3 + y^3 + 3xy(x+y)$,

$$xy = \frac{(x+y)^3 - (x^3 + y^3)}{3(x+y)}$$

so that

$$x^2 + y^2 = (x+y)^2 - 2xy = \frac{(x+y)^3 + 2(x^3+y^3)}{3(x+y)} = \frac{2720}{9}$$

7) **(27)** Since $(3\sqrt{5} - 2i\sqrt{6})^2 = 21 - 12i\sqrt{30}$, the value of $ab + cd$ is 27.

8) **($\sqrt{3c/2}$)** Call the integral $I(b)$. Evaluating yields

$$I(b) = \frac{3ac + a^3 b}{3(b+2)}$$

If the integral is independent of $b$, then $I(b)$ is constant. Thus,

$$\lim_{b \to 0^+} I(b) = \lim_{b \to \infty} I(b)$$

$$\frac{ac}{2} = \frac{a^3}{3}$$

$$a = \sqrt{\frac{3c}{2}}$$

9) ($\cos^{-1}(3/4)$) Let $A$ be the origin and $AB = 1$. Then, $\vec{c} = \frac{3}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$ and $\vec{g} = \frac{3}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{k}$. Using the dot product gives $\vec{c} \cdot \vec{g} = |\vec{c}||\vec{g}| \cos\theta \Rightarrow (3/2)(3/2) + (\sqrt{3}/2)(0) + (0)(\sqrt{3}/2) = \sqrt{3}\sqrt{3}\cos\theta$. When simplified, this gives $\cos\theta = 3/4$, so that $\theta = \cos^{-1}(3/4)$.

10) ($\sqrt{17}$) Note that

$$A^2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = A \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

and also

$$A^2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = A \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

So let

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

and solve the equations to get

$$A = \begin{bmatrix} 2 & 1 & -3 \\ 3 & 1 & -8 \\ 2 & 1 & -4 \end{bmatrix}$$

Then,

$$\mathbf{v} = A^4 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \\ -1 \end{bmatrix}$$

So that the length of $\mathbf{v}$ is $\sqrt{17}$.