Milton Math Tournament

Varsity Individual Written Test

PART I: Multiple Choice

---

1. If g is a function defined on the open interval (a, b) and $a < g < x$ for all $x \in (a, b)$, then which of the following must be true?

a) g is unbounded
b) g is nonconstant
c) g is nonnegative
d) g is monotonic
e) none of the above

---

2. Find the period of the function

$$f(x) = \frac{\sin 2x + \cos 4x}{\csc 2x + \sec 4x}$$

a) π
b) $\frac{\pi}{2}$
c) 2π
d) $\frac{\pi}{4}$
e) none of the above

---

3. Define for $a, b > 0$ the operation $a\S b = a^2 - b\S n$ for a constant n. Given that $16\S 16 = 14$, find $13\S 7$.

a) 17
b) 24
c) 123
d) 106
e) none of the above

---

4. A 3×3 invertible matrix A is inverted and multiplied on the right by a three-dimensional vector. The result is transposed and multiplied on the left by another three-dimensional vector. This result is multiplied on the right by A. When this final product is transposed, what remains?

a) a scalar
b) a three-dimensional vector
c) a 3×3 matrix
d) undefined
e) none of the above

---

5. Find in degrees the smallest positive angle θ such that $\sin 7\theta = \cos 13\theta$.

a) 4.5
b) 9
c) 18
d) 27
e) none of the above

---

6. How many of the following constructions are possible?

1) Given segments of length a and b, construct a segment of length ab
2) Given segments of length a and b, construct a segment of length $\sqrt{ab}$
3) Given segments of length a and 1, construct a segment of length $1/a$
4) Given segments of length a and b, construct a segment of length $a + b$

a) 0
b) 1
c) 2
d) 3
e) 4

---

7. $Q(x)$ and $P(x)$ are polynomials of degree 5. The y-intercept of P is 300 less than that of Q, but the two are equal for $x = 1, 2, 3, 4, 5$. If $Q(7) = -1234$, find $P(7)$.

a) 0
b) 566
c) 3034
d) cannot be determined
e) none of the above

---

8. What is the answer to this question?

a) There is no answer!
b) None of the below
c) None of the above
d) (b) is the answer.
e) Two of the above work.

---

9. Isosceles $\triangle ABC$ is such that $m\angle A = \theta$, $AB = AC = \sqrt{\sin\theta}$, and $BC = \sin\theta$. Find the area of $\triangle ABC$.

a) 1/2
b) 8/25
c) 9/16
d) cannot be determined
e) none of the above

---

10. Let $d(n)$ be the number of divisors of n. Find the value of

$$d\left(d\left(2003^{2003}\right)\right)$$

a) 12
b) 100
c) 2003
d) 2003^2
e) none of the above

---

11. Find the sum of all $x \in (0, \pi)$ such that

$$\csc x = (\sin x)^{(\sin x)^{(\sin x)^{\cdot^{\cdot^{\cdot}}}}}$$

a) π
b) $\pi/2$
c) $\pi/3$
d) $\pi/6$
e) none of the above

---

12. Define $f_k(x) = |x - k|$. Let the solution to $f_k(x) < f_{k+1}(x)$ be $x > a_k$. Find the value of

$$\sum_{k=0}^{100} a_k$$

a) 5000
b) 5050
c) 5100
d) 5150
e) none of the above

---

13. Let $\left\langle \sum a_n \right\rangle$ be the number of terms the series $\sum a_n$ has when expanded. Compute the value of

$$\left\langle \sum_{k=1}^{5} \sum_{j=k}^{k^2} \sum_{m=k}^{j} k^{j^m} \right\rangle$$

a) 40
b) 312
c) 357
d) 400
e) none of the above

---

14. How many points determine an ellipse?

a) 3
b) 4
c) 5
d) 6
e) none of the above

---

15. Caffeinated Joe and Diligent Bob are painting a fence together. Since Joe relies completely on java to get himself moving, his rate of work is variable, whereas Bob works at a constant rate. Joe works at his normal (constant) rate for 3 hours, works at half his rate for an hour, and then twice his normal rate for an hour, after which the cycle repeats itself. Bob by himself would take 12 hours to paint the fence. When Bob works with Joe, they take 8 hours. How many hours would it take Joe by himself?

a) 12
b) 13
c) 17.5
d) 19.5
e) none of the above

---

16. Find the shortest distance from the point $(2, 4)$ to the point $(5, 2)$ that touches both the x and y axes.

a) $\sqrt{61}$
b) $\sqrt{65}$
c) $\sqrt{79}$
d) $\sqrt{85}$
e) none of the above

---

17. Which of the following are equal to $\binom{m+n+p}{r}$?

$$\text{I. } \sum_{k=0}^{r}\sum_{j=0}^{r-k}\binom{m}{k}\binom{n}{j}\binom{p}{r-k-j} \qquad \text{II. } \sum_{k=0}^{r}\binom{m}{k}\binom{n+p}{n+p+k-r}$$

$$\text{III. } \binom{m+n}{r}\binom{p}{r}+\binom{m+p}{r}\binom{n}{r}+\binom{n+p}{r}\binom{m}{r}$$

a) I only
b) II only
c) I and II only
d) II and III only
e) none of the above

---

18. Let A and B be square $n \times n$ matrices such that $ABAB = 0_{n\times n}$. How many of A and B must be invertible so that the matrix $BABA = 0_{n\times n}$ as well?

a) 0
b) 1
c) 2
d) $BABA$ cannot equal $0_{n\times n}$
e) $BABA$ does not rely upon the invertibility of A and B

---

19. Given an arithmetic sequence a_n, define $P_n = a_n a_{n+1} a_{n+2} a_{n+3}$. What is the maximum number of perfect squares that can appear in the sequence P_n when the common difference of a_n is 3?

a) 0
b) 1
c) 2
d) more than 2, but finitely many
e) infinitely many

---

20. Which of the following functions takes a positive number n and transforms it into m^2, where m is the greatest integer such that $m^2 \leq n$?

a) $\sqrt{\lfloor n^2 \rfloor}$
b) $\lfloor \sqrt{n} \rfloor^2$
c) $\lfloor n \rfloor$
d) $\lfloor n + 1/2 \rfloor$
e) none of the above

---

21. Find $F(100)$ if the function $F(k)$ is defined for $k > 1$ as

$$F(k) = \left[\ln\left(1 - \frac{1}{k+1}\right) + \frac{1}{k}\ln k\right]^{-1}\left[\ln\left(\frac{k^{1/k}}{(k+1)^{1/(k+1)}}\right)\right]$$

a) 1
b) 100
c) 1/100
d) 1/101
e) none of the above

---

22. Find the minimum value of

$$(x-y)^2 + (|x| + |y-1| + 1)^2$$

a) 0
b) 1
c) 2
d) 3
e) none of the above

---

23. Find the sum of the coefficients when the trinomial $(x + y - 4)^6$ is expanded completely.

a) 64
b) 128
c) 256
d) 512
e) none of the above

---

24. How many strictly increasing sequences of positive integers begin with 1 and end with 14?

a) 2048
b) 4096
c) 8192
d) 16284
e) none of the above

---

25. Let $a_n = 16, 4, 1, \ldots$ be a geometric sequence. Define P_n as the product of the first n terms. Find the value of

$$\sum_{n=1}^{\infty} \sqrt[n]{P_n}$$

a) 4
b) 8
c) 16
d) 32
e) none of the above

---

26. Find the sum of the absolute values of the roots of the equation

$$(9x^2 - 16)^3 + (16x^2 - 9)^3 = (25x^2 - 25)^3$$

a) 0
b) 1
c) 37/6
d) 5
e) none of the above

---

27. Set A has m elements while set B has n elements. How many different functions $f : A \to B$ exist?

a) m
b) n
c) mn
d) m^n
e) none of the above

---

28. Find the value of

$$\left(\frac{\sin 120^\circ}{16\cos 15^\circ \cos 30^\circ \cos 60^\circ \cos 120^\circ \cos 240^\circ}\right)^2$$

a) $1/2 - \sqrt{3}/4$
b) $1/4 - \sqrt{3}/2$
c) $1/8 - \sqrt{3}/4$
d) $1/2 - \sqrt{3}/2$
e) none of the above

---

29. An ant and a piece of candy are at opposite corners of a hexagon. The ant, in one move, chooses one of the two sides at random and moves along it to the next corner. Then he chooses randomly again (since he is not the most intelligent ant) and so on. What is the expected number of moves for the ant to reach the candy?

a) less than 8
b) 8
c) 9
d) 10
e) more than 10

---

30. Al, Bob, and Carl have a three-way shooting duel. They all know that when Al shoots, he hits someone half of the time, and when Carl shoots, he hits someone 4/5 of the time; they all know that Bob never misses! The duel works as follows: the first shooter is chosen randomly. That person chooses a target and shoots at it. The chance to shoot then moves to the next person ($A \to B \to C \to A$), skipping dead people. The duel ends when two are dead. If everyone takes their optimal strategy, who is most likely to walk away?

a) Al
b) Bob
c) Carl
d) They are all equally likely.
e) Who cares? They're just stupid guys anyway!

---

31. Bob and Carl, having walked away from the duel, decide to celebrate at a local bar. They order a keg of pickle juice, which can be filled with any amount, from 0 (a rip-off, indeed) to 10 pints. If the two guys consume 10 pints, they are guaranteed to be happy; if they get none, they will be unhappy. The probability that they are happy is directly proportional to the fifth power of the amount of juice they consume. So obviously, the happiness of the men is in the hands of the bartender. If he chooses the amount of juice to serve at random, what is the probability the two guys leave happy?

a) 0
b) 1/10
c) 1/5
d) 1/2
e) none of the above

---

32. Let $f_c(x) = \min\{(x-c)^2, (x-c-2)^2\}$. Define $g(c)$ as

$$g(c) = \int_0^1 f_c(x)\,dx$$

What is the minimum value of $g(c)$ for $c \in [-2, 2]$?

a) 0
b) 1/12
c) 1/4
d) 1/3
e) none of the above

---

33. Find z in terms of y if

$$\log_{\sin x} \tan x = y$$

$$(\sin x)^z + \sin^2 x = 1$$

a) $2 - 2y$
b) $2y - 2$
c) $y + 1$
d) $y - 1$
e) none of the above

---

34. The curvature of the curve $y^2 = 2x$ at the point $(2, 2)$ is closest to what integer?

a) 10
b) 11
c) 12
d) 13
e) none of the above

---

35. Greedy Gina has a rich uncle. Gina's uncle knows of his niece's greed and decides to test her mettle. Thus he makes this proposition:

> "Gina, as you know, this is day 0 of a month that goes to day 10. You choose an integer n from 1 to 10, inclusive. I will take that number and pay you $(10-n)^t$ each day, where t is the current day (0, 1, 2, etc. up to 10). But there is a catch! Everyday that is a multiple of n, I will take back all the money that you currently have!"

To help her determine the best n for maximizing her bank account, Gina writes a program in C++. Her program is on the next page. For how many values of n will Gina's program return the incorrect result?

a) 0
b) 1
c) 2
d) 10
e) none of the above

[program for question 35]

```
#include <iostream.h>
#include <math.h>

int dayAmount(int day, int number);
void updateSum(int day, int number, int sum);
int totalAmount(int number);

int main()
{
    int number;
    cout << "What is n?" << endl;
    cin >> number;
    cout << "Your final amount will be $ " << totalAmount(number);
    return 0;
}

int dayAmount(int day, int number)
{
    return pow(10 - number, day);
}

void updateSum(int day, int number, int sum)
{
    if (day % number == 0)
      sum = 0;
    else
      sum += dayAmount(day, number);
}

int totalAmount(int number)
{
    int day, sum = 0;

    for(day = 0; day <= 10; day++)
    {
        updateSum(day, number);
    }

    return sum;
}
```

1. Given that P, Q, R, S are statements with random truth values (either true or false), find the probability that the following statement is true:

$$\Big(P \vee \big[(Q \vee P) \to (R \vee S)\big]\Big) \wedge \big(S \to [R \wedge Q]\big)$$

---

2. The sequence 1, 2, 2, 3, 3, 3, 4, 4, 4, 4,... is such that the integer n appears exactly n times. Find the 2003rd term of this sequence.

---

3. A student has 37 days to prepare for an exam. She needs no more than 60 hours of study, but she wishes to study at least 1 hour per day. No matter how she schedules her study, there is a succession of consecutive days where she will have studied exactly $r \geq 10$ hours. Find r, assuming that she only studies in whole integer numbers of hours.

---

4. Find the minimum positive integer a such that

$$5x^7 + 7x^5 + ax$$

is divisible by 35 for all integers x.

---

5. The equation

$$\frac{\cos 2\theta - \cos 10\theta}{\sin 2\theta + \sin 10\theta} = \tan k\theta$$

is an identity. Find the value of k.

---

6. Given that $x + y = 30$ and $x^3 + y^3 = 100$, find $x^2 + y^2$.

---

7. If the expression

$$\sqrt{21 - 12i\sqrt{30}}$$

is written in the form $a\sqrt{b} - ci\sqrt{d}$ (with b and d not containing the square of an integer), find $ab + cd$.

---

8. Let a, b, and c be positive constants. Find a in terms of c if the value of the integral

$$\int_0^1 (acx^{b+1} + a^3bx^{3b+5})dx$$

is independent of b.

---

9. $ABCDEF$ and $ABGHIJ$ are regular hexagons in perpendicular planes. Find $m\angle CAG$.

---

10. The matrix A satisfies

$$A \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

$$A^2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$A^3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Find the length of the vector $\mathbf{v}$, where

$$\mathbf{v} = A^4 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

END OF WRITTEN TEST
