JV Power Question

Let n, k and m be positive integers. Define $f_k(x)$ and $S(x)$ as:

$$f_k(x) = |x - k|$$

$$S(x) = \sum_{k=1}^{n} f_k(x)$$

I.
If $x \leq m < n$, determine, without proof, which of the following must be true.
a) $f_m(x) = m - x$
b) $f_j(x) = j - x$ where $j > m$
c) $f_j(x) = x - j$ where $j < m$
d) $S(x) > 0$

II.
a) If $n = 2$, find $S(31/17)$.
b) If $n = 3$, solve $S(x) = x$.
c) If $x \leq 1$, find a simplified form of $S(x)$ (involving no sigma).
d) If $x \geq n$, find a simplified form of $S(x)$.
e) Now assume that $0 < m - 1 \leq x \leq m \leq n$. Show that

$$S(x) = (2x - m)(m - 1) + n\left(\frac{n+1}{2} - x\right)$$

III.
a) Let n be odd. Show that $S(x)$ attains a minimum value at $x = \frac{n+1}{2}$ and find that minimum value.
b) Let n be even. Show that $S(x)$ attains a minimum value everywhere on the interval $[\frac{n}{2}, 1 + \frac{n}{2}]$ and find that minimum value.

Consider the equation

$$S(x) = m - m^2 + \frac{n(n+1)}{2}$$

c) Find a lower bound on n in terms of m if the equation is satisfied by some $x \leq 1$.
d) Find a lower bound on n in terms of m if the equation is satisfied by some $x \geq n$.
e) Prove that, if $n = 2(m - 1)$, then the equation is satisfied by some $x \in (1, n)$.