I.

1. To find $P(A)$, we consider the combinations of dice A and B that result in A's victory. We make a chart:

Die A	Die B	Probability
9	3	1/25
9	8	1/25
9	2	1/10
20	3	1/25
20	11	1/50
20	8	1/25
20	2	1/10

The sum of the probabilities give $P(A) = 19/50$. Then $P(B) = 1 - P(A) = 31/50$. Since $P(B) > P(A)$, you should choose die B.

2. For the game to be fair,

$$45 \cdot P(A) = m \cdot P(B),$$

where m represents the return from a win with die B. Thus

$$m = \frac{45 \cdot P(A)}{P(B)} = \frac{45 \cdot P(A)}{1 - P(A)}$$

Since $P(A) = 19/50$, $m = 855/31 \approx 27.58$.

3. Note that if two dice match, you lose. The probability of this not happening is $\frac{19}{20} \cdot \frac{18}{20}$. If it doesn't happen, then the probability of winning is $1/3$. Thus the chance of winning is $\frac{(19)(18)}{(20)(20)(3)} = 57/200$. The expected value of the game is given by

$$E = 56 = -1 + \left(\frac{57}{200}\right) w$$

where w represents the amount won per game. Solving this equation yields $w = 200$. Therefore, you should \$200 each time you win.

II.

1. A win multiplies your balance by $(1 + f)$ whereas a loss multiplies it by $(1 - f)$. Since there are w wins there are $(n - w)$ losses; thus we have

$$x_n = x_0(1 + f)^w (1 - f)^{n-w}$$

2. Since $y^n = (1 + f)^w (1 - f)^l$ (where $l = n - w$), we have

$$y(f) = (1 + f)^{w/n} (1 - f)^{l/n}$$

We can define $g(f)$ as

$$g(f) = \ln y = \frac{1}{n}\Big[w\ln(1+f) + l\ln(1-f)\Big]$$

Note that the maximum of g occurs at the same value of f as the maximum of y. We rearrange a little, letting $P = w/n$ be the probability of winning and $Q = 1 - P$ be the probability of losing:

$$g(f) = P\ln(1+f) + Q\ln(1-f)$$

Taking the derivative of this and setting equal to 0 yields

$$g'(f) = \frac{P}{1+f} - \frac{Q}{1-f} = 0$$

$$\frac{P}{1+f} = \frac{Q}{1-f}$$

$$f = \frac{P-Q}{P+Q} = P - Q$$

Setting $P = \frac{p}{p+1}$ yields

$$f = \frac{p-1}{p+1}$$

3. Since the game is biased in your favor, $0 < E = Pa - Qb$. Therefore, $Pa > Qb$ which yields $a/b > Q/P$. Recall that $P = \frac{p}{p+1}$ and $Q = 1 - P = \frac{1}{p+1}$. Then

$$\frac{b}{a} < \frac{\frac{p}{p+1}}{\frac{1}{p+1}} = p$$

4. A win under the new system multiplies your balance by $(1 + af)$ while a loss multiplies your balance by $(1 - bf)$. Therefore,

$$g(f) = P\ln(1+af) + Q\ln(1-bf)$$

$$g'(f) = \frac{Pa}{1+af} - \frac{Qb}{1-bf} = 0$$

$$f = \frac{Pa - Qb}{ab} = \frac{1}{p+1}\left(\frac{ap-b}{ab}\right)$$

III.

1. The casino's cut reduces the pool to $(1-\alpha)N$. Since N_i is the total amount bet on horse i, d_n/N_i is the proportion of money that the bettor gets from the remaining pool $(1-\alpha)N$. Therefore, the bettor receives

$$R_{n,i} = (1-\alpha)\frac{Nd_n}{N_i}$$

2. Since a winning bet of \$1 yields $R_{n,i}$ in return, A winning bet of $1/R_{n,i}$ yields \$1 in return. If a bettor bets this amount on each horse, his total cost is

$$\sum_{i=1}^{h} {R_{n,i}}^{-1}$$

If this cost is less than one, the bettor turns a profit *guaranteed*, since he is sure to win with one ticket an amount of \$1. That isn't exactly fair!

3. Using the formula for $R_{n,i}$,

$$\sum_{i=1}^{h} {R_{n,i}}^{-1} = \sum_{i=1}^{h} \frac{N_i}{(1-\alpha)Nd_n} = \frac{\sum_{i=1}^{h} N_i}{(1-\alpha)Nd_n}$$

Since $\sum_{i=1}^{h} N_i = N$ and $d_n = 1$, the expression becomes

$$\sum_{i=1}^{h} {R_{n,i}}^{-1} = \frac{N}{(1-\alpha)N} = \frac{1}{(1-\alpha)}$$

Since $0 < \alpha < 1$,

$$\frac{1}{1-\alpha} > 1$$

Therefore,

$$\sum_{i=1}^{h} {R_{n,i}}^{-1} > 1$$

4. For a given race, the profit, P, is the difference between $R_{n,i}$ and hd (the money required to bet d dollars on each of h horses). So, under the new system,

$$P = (1-\alpha)\frac{Nd}{N_i} - d(h-1)$$

We require that $P > 0$. Since you are the only bettor for the race, $N = hd$ and $N_i = d$. Using these facts,

$$(1-\alpha)\frac{Nd}{d} - d(h-1) > 0$$
$$(1-\alpha)N > N_i(h-1)$$
$$N - N\alpha > hN_i - N_i$$
$$dh - \alpha dh > hd - d$$
$$\alpha h < 1$$

Therefore, $h < 1/\alpha$.