Milton Math Tournament

Varsity Power Question : Probability

You attend the grand opening of the new casino 'Chuck's Paradise.' The casino (or *house* in gambling terminology) has brought in massive amounts of money for the visiting high rollers—so much in fact, that the monetary supply is infinite. You, on the other hand, have a measly x_0 dollars as your balance.

I. *It slices, it dices ...*

The first game you come to is a dice game that uses dice in the shapes of icosahedrons. Die A has four 9's, two 0's, ten 1's, and four 20's on its faces while die B has four 3's, two 11's, four 8's, and ten 2's on its faces. All faces have an equal chance of landing on top. The player chooses a die and rolls it. Then the house rolls the remaining die. Whoever has the higher number wins.

1. Which die should you choose to maximize your odds of winning?

2. Suppose winning with die A yields \$45. How much should winning with die B yield in order for the game to be fair?

3. At a different table, the house has two icosahedron dice, each with all of the numbers $1-20$. The player gets one die numbered the same way. The house rolls its dice and the player rolls his. For the player to win, his number must be *strictly* between the numbers the house rolled (ties result in your loss). If each game costs one dollar, and you can be expected to walk away from the table with \$56 more than you started with, how much does winning pay off per game?

II. *Fun with coins*

You leave the casino and visit the bank to deposit your large wad o' cash, thus leaving you with only x_0 dollars in your pocket. Upon returning, you find a coin toss game. At the coin toss game, the rules are simple: heads yields a win, tails yields a loss. You can bet any amount that you wish up to your current balance. However, the coin is biased in your favor! The odds of winning are p to 1 ($p > 1$). As a careful gambler, you bet a constant fraction f ($0 < f < 1$) of your current balance each time you play. Give all answers in this section in terms of the given values.

1. Define the sequence $\{x_i\}$ as your balance after i plays of the coin toss game. If you win w times, find $\{x_n\}$ in terms of x_0.

2. After n plays, you wish to approximate y, the *average increase per play.* Symbolically, you write

$$y^n = x_n/x_0.$$

Using the answer from (1) above, explicitly define $y(f)$ and find the value of f for which y is maximized (this value of f yields the maximum value of $E(x_n)$, where E is expected value).

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After your amazing victories via the strategy of (2), the house changes things up. Now, when you win, you receive a dollars for every dollar bet. Likewise, a loss takes b dollars for every dollar bet.

3. Show that the odds are in your favor if and only if $b/a < p$.

4. Find the new fraction f that you should bet each game to maximize your balance.

III. *A night at the races*
After spending the money you earned on a glass of water, you find yourself back at x_0 dollars yet again. You return to exploring the casino (eager to find a more interesting game) and come across a gigantic racetrack for horses. Every 15 minutes, a race occurs consisting of h horses. Only one type of bet exists at this track: bet an amount of your choice (up to your current balance, as always) on which horse will win. You can make as many bets per race as you wish. Let N_i be the total bets that are made on the ith horse. Then the total 'pool' for the race is $N = \sum N_i$. After a race, the casino removes a proportion α $(0 < \alpha < 1)$ of the pool and divides the remaining money among the winners, though not equally—a bettor's return is proportional to the amount he bet. The individual return $(R_{n,i})$ is the winnings received by the nth winner for betting on the ith horse.

1. Show that, if the ith horse wins,

$$R_{n,i} = (1 - \alpha)\frac{Nd_n}{N_i}$$

where d_n is the nth bettor's bet.

2. If a \$1 bet yields a return of $R_{n,i}$, explain why

$$\sum_{i=1}^{h} {R_{n,i}}^{-1} < 1$$

would be unfair to the house.

3. Show that the condition of (2) cannot happen at the casino's racetrack.

4. For the rest of the night, the casino holds a special: bettors are reimbursed for buying winning tickets. In other words,

$$R_{n,i} = (1 - \alpha)\frac{Nd_n}{N_i} + d_n$$

You notice that nobody is betting on the next race, so, in an attempt to pull an easy profit, you put \$$d$ on each of the h horses. You know, however, that whether or not your plan will work depends on the value of h. Show that you can turn a profit if and only if $h < 1/\alpha$.