JV Power Solutions

I.
a) When $x - m \leq 0$, by definition, $f_m(x) = |x - m| = -(x - m) = m - x$. Thus the statement is **true**.

b) Since $j > m$ and $m > x$ implies that $j > x$, by the same logic as above, the statement is **true**.

c) When both j and x are less than m, there is no way to know the relation between x and j. Therefore the statement is **false**.

d) Let $x = m = n = 1$. This yields S(x) = 0. Therefore the statement is **false**.

II.
a) Note that $1 < 31/17 < 2$. Then, for $1 < x < 2$ we have

$$S(x) = f_1(x) + f_2(x)$$

$$S(x) = |x - 1| + |x - 2|$$

$$S(x) = (x - 1) + (2 - x)$$

$$S(x) = 1$$

Thus $S(31/17) = 1$.

b) First we find an expression for $S(x)$ when $n = 3$. We divide the number line into four intervals: $x \leq 1$, $1 < x < 2$, $2 < x < 3$, and $x \geq 3$. For the first interval,

$$S(x) = (1 - x) + (2 - x) + (3 - x) = 6 - 3x$$

Likewise for the others,

$$S(x) = (x - 1) + (2 - x) + (3 - x) = 4 - x$$

$$S(x) = (x - 1) + (x - 2) + (3 - x) = x$$

$$S(x) = (x - 1) + (x - 2) + (x - 3) = 3x - 6$$

Then we graphically check the intersection of S and x in each interval. The only expression that touches x in its interval is x. Therefore the solution occurs wherever $S(x)$ is simplified to x. The solution is $2 < x < 3$.

c) When $x \leq 1$, $f_k(x) = k - x$ for all k, since $k \geq 1$. Therefore, $S(x)$ can be written

$$S(x) = \sum_{k=1}^{n} f_k(x)$$

$$S(x) = \sum_{k=1}^{n}(k - x)$$

$$S(x) = \sum_{k=1}^{n} k - \sum_{k=1}^{n} x$$

$$S(x) = \frac{n(n+1)}{2} - nx$$

d) When $x \geq n$, $f_k(x) = x - k$ for all k, since $k \leq n$. Therefore, $S(x)$ can be written

$$S(x) = \sum_{k=1}^{n} f_k(x)$$

$$S(x) = \sum_{k=1}^{n}(x - k)$$

$$S(x) = \sum_{k=1}^{n} x - \sum_{k=1}^{n} k$$

$$S(x) = nx - \frac{n(n+1)}{2}$$

e) When $0 \leq m - 1 \leq x \leq m$, $f_k(x) = x - k$ for $k \leq m - 1$. Likewise, $f_k(x) = k - x$ for $k \geq m$. Therefore, $S(x)$ can be written

$$S(x) = \sum_{k=1}^{n} f_k(x)$$

$$S(x) = \sum_{k=1}^{m-1}(x - k) + \sum_{k=m}^{n}(k - x)$$

$$S(x) = \sum_{k=1}^{m-1}(x - k) + \sum_{k=1}^{n}(k - x) - \sum_{k=1}^{m-1}(k - x)$$

$$S(x) = 2\sum_{k=1}^{m-1}(x - k) + \sum_{k=1}^{n}(k - x)$$

$$S(x) = \left[2x(m-1) - 2\frac{m(m-1)}{2}\right] + \left[\frac{n(n+1)}{2} - xn\right]$$

$$S(x) = (2x - m)(m - 1) + n\left(\frac{n+1}{2} - x\right)$$

III.

a) Rearrange the formula from part (II e).

$$S(x) = x\Big(2(m-1) - n\Big) + \left[\frac{n(n+1)}{2} - m(m-1)\right]$$

It can be seen that on the interval $m-1 \le x \le m$, $S(x)$ is linear, with a slope of $\Big(2(m-1)-n\Big)$. We can see that small values of x (and thus small values of m) yield a negative value for the slope. In fact, the slope is always negative until $x = (n+1)/2$. To see this, let

$$\frac{n-1}{2} \le x \le \frac{n+1}{2}$$

Since $m - 1 = (n-1)/2$, the slope is -1. When

$$\frac{n+1}{2} \le x \le \frac{n+3}{2}$$

$m - 1 = (n+1)/2$, and so the slope is now 1.

As x increases, $m-1$ increases, and so the slope stays positive (i.e., $S(x)$ only grows). Likewise, when x decreases, $m - 1$ must decrease along with the slope. As $S(x)$ is decreasing when $x < (n+1)/2$ and increasing when $x > (n+1)/2$, $S(x)$ has a minimum at $x = (n+1)/2$.

Now let $x = m = (n+1)/2$. Using the formula from (II e),

$$S\left(\frac{n+1}{2}\right) = \left(n + 1 - \frac{n+1}{2}\right)\left(\frac{n+1}{2} - 1\right) + n\left(\frac{n+1}{2} - \frac{n+1}{2}\right)$$

$$S\left(\frac{n+1}{2}\right) = \frac{n^2-1}{4}$$

b) Again we use the rearranged form of $S(x)$:

$$S(x) = x\Big(2(m-1) - n\Big) + \left[\frac{n(n+1)}{2} - m(m-1)\right]$$

The slope of $S(x)$ is $\Big(2(m-1)-n\Big)$. When $m-1 < n/2$, the slope is negative. When $m-1 > n/2$, the slope is positive. By the reasoning above, the minimum must occur when $m - 1 = n/2$ (and the slope is 0). In other words, the minimum occurs when

$$\frac{n}{2} \le x \le \frac{n}{2} + 1$$

which is what we wanted to show. To find the value of $S(x)$ in this interval, note that the slope is 0, and thus the function is constant on $[n/2, (n+2)/2]$; so plugging in any value on that interval will do the trick. We'll use $x = m = n/2$.

$$S(n/2) = (n - n/2)(n/2 - 1) + n\left(\frac{n+1}{2} - n/2\right)$$

$$S(x) = \frac{n^2}{4}$$

c) Substitute for S(x) the expression you found in part (II c). Then

$$\frac{n(n+1)}{2} - nx = m - m^2 + \frac{n(n+1)}{2}$$

$$x = \frac{m^2 - m}{n}$$

Since $x \leq 1$,

$$\frac{m^2 - m}{n} \leq 1$$

$$m^2 - m \leq n$$

Therefore the lower bound on n is $m^2 - m$.

d) Substitute for S(x) the expression you found in part (II d). Then

$$nx - \frac{n(n+1)}{2} = m - m^2 + \frac{n(n+1)}{2}$$

$$x = \frac{m - m^2}{n} + n + 1$$

Since $x \geq n$,

$$\frac{m - m^2}{n} + n + 1 \geq n$$

$$\frac{m - m^2}{n} \geq -1$$

$$m - m^2 \geq -n$$

$$m^2 - m \leq n$$

Thus the lower bound on n is $m^2 - m$.

e) When $x \leq 1$, $S(x) \geq (n^2 - n)/2$. Likewise, when $x \geq n$, $S(x) \geq (n^2 - n)/2$. Since $S(x)$ is a sum of absolute values, and absolute value functions are continuous, we can conclude that $S(x)$ is continuous as well. Note that, since $n = 2(m-1)$, n is even. By the result of part (III b), $n^2/4$ is a part of the range of $S(x)$. Thus we can conclude that

$$S(x) \in \left[\frac{n^2}{4}, \frac{n^2 - n}{2}\right]$$

implies that $x \in (1, n)$. We now only need to prove that

$$m - m^2 + \frac{n(n+1)}{2} \in \left[\frac{n^2}{4}, \frac{n^2 - n}{2}\right]$$

to show that $x \in (1, n)$. Thus,

$$\frac{n^2}{4} \leq m - m^2 + \frac{n(n+1)}{2} \leq \frac{n^2 - n}{2}$$

$$\frac{-n^2}{4} \leq m - m^2 + \frac{n}{2} \leq \frac{-n}{2}$$

$$n^2 \geq 4(m^2 - m) - 2n \geq 2n$$

Substitute $2(m-1)$ for n.

$$4m^2 - 8m + 4 \geq 4m^2 - 4m - 4m + 4 \geq 4m - 4$$

$$4 \geq 4 \geq (4m - 4) - (4m^2 - 8m)$$

$$4 \leq 4m^2 - 12m - 4$$

$$0 \leq m^2 - 3m + 2$$

$$0 \leq (m-1)(m-2)$$

Which is true for all positive integers m.
Therefore, when $n = 2(m-1)$,

$$m - m^2 + \frac{n(n+1)}{2} \in \left[\frac{n^2}{4}, \frac{n^2 - n}{2}\right]$$

and so the equation $S(x) = m - m^2 + n(n+1)/2$ is satisfied by some $x \in (1, n)$.